A new efficient dynamic programming procedure "forward against forward"

Analysis of multidimensional data arrays

Estimation of additive noise dispersion in the model of non-stationary linear regression.

Data model: $y_t^0 = \mathbf{x}_t^T \mathbf{y}_t + \xi_t$ Experimental data: $(y_t^0, \mathbf{y}_t), t = 1,...,N$

Sought-for parameters: $\mathbf{x}_t \in \mathrm{R}^n$, $t = 1,...,N$, and $D = M(\xi_t^2)$

"Naive" estimation of additive noise dispersion: $\hat{D} = \frac{1}{N}\sum_{t=1}^{N}(y_t^0 - \hat{\mathbf{x}}_t^T \mathbf{y}_t)^2$

Correct estimation of additive noise dispersion: $\hat{D} = \frac{1}{N}\sum_{t=1}^{N}\left(y_t^0 - (\hat{\mathbf{x}}_t^{(t)})^T \mathbf{y}_t\right)^2$

$\hat{\mathbf{x}}_t^{(t)}$ – parameter of non-stationary linear regression model inferred from incomplete sequence of observations $(y_1^0,...,y_{t-1}^0,\ y_{t+1}^0,...,y_N^0)$, where one observation y_t^0 is skipped

Objective function to be minimized for calculating one parameter of non-stationary linear regression

$$\hat{X}^{(t)} = (\hat{\mathbf{x}}_1^{(t)},...,\hat{\mathbf{x}}_N^{(t)}) = \arg\min_{\mathbf{x}_1,...,\mathbf{x}_N} J^{(t)}(\mathbf{x}_1,...,\mathbf{x}_N),$$

$$J^{(t)}(\mathbf{x}_1,...,\mathbf{x}_N) = \sum_{s=1,\, s\neq t}^{N}(y_s^0 - \mathbf{x}_s^T \mathbf{y}_s)^2 + \sum_{s=2}^{N}(\mathbf{x}_{s-1} - \mathbf{x}_s)^T \mathbf{U}_s (\mathbf{x}_{s-1} - \mathbf{x}_s).$$

Left and right Bellman functions

Left partial objective functions and Bellman functions:

$$J_t^-(x_1,\ldots,x_t) = \sum_{s=1}^{t} \psi_s(x_s) + \sum_{s=2}^{t} \gamma_s(x_{s-1}, x_s), \qquad \tilde{J}_t^-(x_t) = \min_{x_1,\ldots,x_{t-1}} J_t^-(x_1,\ldots,x_t)$$

Right partial objective functions and Bellman functions:

$$J_t^+(x_t,\ldots,x_N) = \sum_{s=t}^{N} \psi_s(x_s) + \sum_{s=t}^{N-1} \gamma_{s+1}(x_s, x_{s+1}), \qquad .\tilde{J}_t^+(x_t) = \min_{x_{t+1},\ldots,x_N} J_t^+(x_t,\ldots,x_N)$$

Recurrent relations for Bellman functions

$$\tilde{J}_t^-(x_t) = \psi_t(x_t) + \min_{x_{t-1}} \left[\gamma_t(x_{t-1}, x_t) + \tilde{J}_{t-1}^-(x_{t-1}) \right]$$

$$\tilde{J}_t^+(x_t) = \psi_t(x_t) + \min_{x_{t+1}} \left[\gamma_{t+1}(x_t, x_{t+1}) + \tilde{J}_{t+1}^+(x_{t+1}) \right]$$

Representation of the objective function via Bellman function

$$J(X)=\sum_{s=1}^{N}\psi_s(x_s)+\sum_{s=2}^{N}\gamma_s(x_{s-1},x_s)=$$

$$=\sum_{s=1}^{t-1}\psi_s(x_s)+\sum_{s=2}^{t-1}\gamma_s(x_{s-1},x_s)+\gamma_t(x_{t-1},x_t)+\psi_t(x_t)+\gamma_{t+1}(x_t,x_{t+1})+\sum_{s=t+1}^{N}\psi_s(x_s)+\sum_{s=t+1}^{N-1}\gamma_{s+1}(x_s,x_{s+1})=$$

$$=J_{t-1}^{-}(x_1,\ldots,x_{t-1})+\gamma_t(x_{t-1},x_t)+\psi_t(x_t)+\gamma_{t+1}(x_t,x_{t+1})+J_{t+1}^{+}(x_{t+1},\ldots,x_N)$$

Marginal node functions for inner nodes

$$\hat{J}_t(x_t)=\min_{x_s,\,s\in T,\,s\neq t}J(X)=$$

$$=\min_{x_1,\ldots,x_{t-1},x_{t+1},\ldots,x_N}\left(J_{t-1}^{-}(x_1,\ldots,x_{t-1})+\gamma_t(x_{t-1},x_t)+\psi_t(x_t)+\gamma_{t+1}(x_t,x_{t+1})+J_{t+1}^{+}(x_{t+1},\ldots,x_N)\right)=$$

$$=\min_{x_1,\ldots,x_{t-1}}\left(J_{t-1}^{-}(x_1,\ldots,x_{t-1})+\gamma_t(x_{t-1},x_t)\right)+\psi_t(x_t)+\min_{x_{t+1},\ldots,x_N}\left(\gamma_{t+1}(x_t,x_{t+1})+J_{t+1}^{+}(x_{t+1},\ldots,x_N)\right)=$$

$$=\min_{x_{t-1}}\left[\gamma_t(x_{t-1},x_t)+\min_{x_1,\ldots,x_{t-2}}J_{t-1}^{-}(x_1,\ldots,x_{t-1})\right]+\psi_t(x_t)+\min_{x_{t+1}}\left[\gamma_{t+1}(x_t,x_{t+1})+\min_{x_{t+2},\ldots,x_N}J_{t+1}^{+}(x_{t+1},\ldots,x_N)\right]$$

$$=\min_{x_{t-1}}\left[\gamma_t(x_{t-1},x_t)+\tilde{J}_{t-1}^{-}(x_{t-1})\right]+\psi_t(x_t)+\min_{x_{t+1}}\left[\gamma_{t+1}(x_t,x_{t+1})+\tilde{J}_{t+1}^{+}(x_{t+1})\right]$$

Dynamic programming procedure "forward against forward"

Forward pass along the signal from the left to the right $t = 1,\ldots,N$. Left Bellman functions are to be calculated and stored:

$$\tilde{J}_1^- = \psi_1(x_1),$$
$$\tilde{J}_t^-(x_t) = \psi_t(x_t) + \min_{x_{t-1}}\left[\gamma_t(x_{t-1}, x_t) + \tilde{J}_{t-1}^-(x_{t-1})\right].$$

Forward pass along the signal from the right to the left $t = N,\ldots,1$. Right Bellman functions are to be calculated and stored:

$$\tilde{J}_N^+ = \psi_N(x_N),$$
$$\tilde{J}_t^+(x_t) = \psi_t(x_t) + \min_{x_{t+1}}\left[\gamma_{t+1}(x_t, x_{t+1}) + \tilde{J}_{t+1}^+(x_{t+1})\right].$$

Joining pass. At each node, the marginal function is calculated from the respective left and right Bellman functions:

$$\hat{J}_1(x_1) = \tilde{J}_1^+(x_1) = \psi_1(x_1) + \min_{x_2}\left[\gamma_2(x_1, x_2) + \tilde{J}_2^+(x_2)\right],$$
$$\hat{J}_t(x_t) = \min_{x_{t-1}}\left[\gamma_t(x_{t-1}, x_t) + \tilde{J}_{t-1}^-(x_{t-1})\right] + \psi_t(x_t) + \min_{x_{t+1}}\left[\gamma_{t+1}(x_t, x_{t+1}) + \tilde{J}_{t+1}^+(x_{t+1})\right],$$
$$\hat{J}_N(x_N) = \tilde{J}_N^-(x_N) = \psi_N(x_N) + \min_{x_{N-1}}\left[\gamma_N(x_{N-1}, x_N) + \tilde{J}_{N-1}^-(x_{N-1})\right].$$

Parametric quadratic procedure "forward against forward"

Quadratic node functions: $\psi_t(\mathbf{x}_t) = b_t^0 + (\mathbf{x}_t - \mathbf{x}_t^0)^T \mathbf{Q}_t^0 (\mathbf{x}_t - \mathbf{x}_t^0)$

Quadratic edge functions: $\gamma_t(\mathbf{x}_{t-1}, \mathbf{x}_t) = (\mathbf{x}_{t-1} - \mathbf{A}\mathbf{x}_t)^T \mathbf{U}_t (\mathbf{x}_{t-1} - \mathbf{A}\mathbf{x}_t)$

$$\gamma_t(\mathbf{x}_{t-1}, \mathbf{x}_t) = (\mathbf{x}_t - \mathbf{A}^{-1}\mathbf{x}_{t-1})^T \mathbf{A}^T \mathbf{U}_t \mathbf{A} (\mathbf{x}_t - \mathbf{A}^{-1}\mathbf{x}_{t-1})$$

Left Bellman functions:

$$\tilde{J}_t^-(\mathbf{x}_t) = \tilde{b}_t^- + (\mathbf{x}_t - \tilde{\mathbf{x}}_t^-)^T \tilde{\mathbf{Q}}_t^- (\mathbf{x}_t - \tilde{\mathbf{x}}_t^-)$$

$$\tilde{\mathbf{Q}}_t^- = \mathbf{Q}_t^0 + \mathbf{A}^T \left((\tilde{\mathbf{Q}}_{t-1}^-)^{-1} + \mathbf{U}_t^{-1} \right)^{-1} \mathbf{A},$$

$$\tilde{\mathbf{x}}_t^- = (\tilde{\mathbf{Q}}_t^-)^{-1} \left\{ \mathbf{Q}_t^0 \mathbf{x}_t^0 + \mathbf{A}^T \left((\tilde{\mathbf{Q}}_{t-1}^-)^{-1} + \mathbf{U}_t^{-1} \right)^{-1} \tilde{\mathbf{x}}_{t-1}^- \right\},$$

$$\tilde{b}_t^- = b_t^0 + \tilde{b}_{t-1}^- + (\mathbf{x}_t^0 - \tilde{\mathbf{x}}_t^-)^T \mathbf{Q}_t^0 \mathbf{x}_t^0 + \left(\tilde{\mathbf{x}}_{t-1}^- - \mathbf{A}\mathbf{x}_t \right)^T \left((\tilde{\mathbf{Q}}_{t-1}^-)^{-1} + \mathbf{U}_t^{-1} \right)^{-1} \tilde{\mathbf{x}}_{t-1}^-,$$

Right Bellman functions:

$$\tilde{J}_t^+(\mathbf{x}_t) = \tilde{b}_t^+ + (\mathbf{x}_t - \tilde{\mathbf{x}}_t^+)^T \tilde{\mathbf{Q}}_t^+ (\mathbf{x}_t - \tilde{\mathbf{x}}_t^+)$$

$$\tilde{\mathbf{Q}}_t^+ = \mathbf{Q}_t^0 + (\mathbf{A}^{-1})^T \left((\tilde{\mathbf{Q}}_{t+1}^+)^{-1} + (\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} \right)^{-1} \mathbf{A}^{-1},$$

$$\tilde{\mathbf{x}}_t^+ = (\tilde{\mathbf{Q}}_t^+)^{-1} \left\{ \mathbf{Q}_t^0 \mathbf{x}_t^0 + (\mathbf{A}^{-1})^T \left((\tilde{\mathbf{Q}}_{t+1}^+)^{-1} + (\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} \right)^{-1} \tilde{\mathbf{x}}_{t+1}^+ \right\},$$

$$\tilde{b}_t^+ = b_t^0 + \tilde{b}_{t+1}^+ + (\mathbf{x}_t^0 - \tilde{\mathbf{x}}_t^+)^T \mathbf{Q}_t^0 \mathbf{x}_t^0 + \left(\tilde{\mathbf{x}}_{t+1}^+ - \mathbf{A}^{-1}\mathbf{x}_t \right)^T \left((\tilde{\mathbf{Q}}_{t+1}^+)^{-1} + (\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} \right)^{-1} \tilde{\mathbf{x}}_{t+1}^+,$$

Marginal quadratic node functions:

$$\hat{J}_t(\mathbf{x}_t) = \hat{b} + (\mathbf{x}_t - \hat{\mathbf{x}}_t)^T \hat{\mathbf{Q}}_t (\mathbf{x}_t - \hat{\mathbf{x}}_t)$$

$$\hat{\mathbf{Q}}_t = \left(\mathbf{U}_t^{-1} + (\tilde{\mathbf{Q}}_{t-1}^{-})^{-1}\right)^{-1} + \mathbf{Q}_t^0 + \left((\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} + (\tilde{\mathbf{Q}}_{t+1}^{+})^{-1}\right)^{-1},$$

$$\hat{\mathbf{x}}_t = \hat{\mathbf{Q}}_t^{-1}\left[\left(\mathbf{U}_t^{-1} + (\tilde{\mathbf{Q}}_{t-1}^{-})^{-1}\right)^{-1} \tilde{\mathbf{x}}_{t-1}^{-} + \mathbf{Q}_t^0 \mathbf{x}_t^0 + \left((\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} + (\tilde{\mathbf{Q}}_{t+1}^{+})^{-1}\right)^{-1} \tilde{\mathbf{x}}_{t+1}^{+}\right],$$

$$\hat{b} = \tilde{b}_t^{-} + b_t^0 + \tilde{b}_t^{+} + (\tilde{\mathbf{x}}_{t-1}^{-} - \mathbf{A}\mathbf{z})^T \left(\mathbf{U}_t^{-1} + (\tilde{\mathbf{Q}}_{t-1}^{-})^{-1}\right)^{-1} \tilde{\mathbf{x}}_{t-1}^{-} + (\mathbf{x}_t^0 - \mathbf{z})^T \mathbf{Q}_t^0 \mathbf{x}_t^0 +$$

$$(\tilde{\mathbf{x}}_{t+1}^{+} - \mathbf{A}^{-1}\mathbf{z})^T \left((\mathbf{A}^T \mathbf{U}_{t+1} \mathbf{A})^{-1} + (\tilde{\mathbf{Q}}_{t+1}^{+})^{-1}\right)^{-1} \tilde{\mathbf{x}}_{t+1}^{+}.$$

Evaluation of smoothness degree

Model of non-stationary regression

$$y_t^0 = \mathbf{x}_t^T \mathbf{y}_t + \xi_t$$

Quadratic penalty on unsmoothness of regression coefficients:

$$\gamma_t(\mathbf{x}_{t-1}, \mathbf{x}_t) = (\mathbf{x}_{t-1} - \mathbf{x}_t)^T \mathbf{U}_t (\mathbf{x}_{t-1} - \mathbf{x}_t),$$

$$\mathbf{U}_t = \mathbf{U} = \begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & u \end{pmatrix}, \qquad u \text{ – smoothness degree.}$$

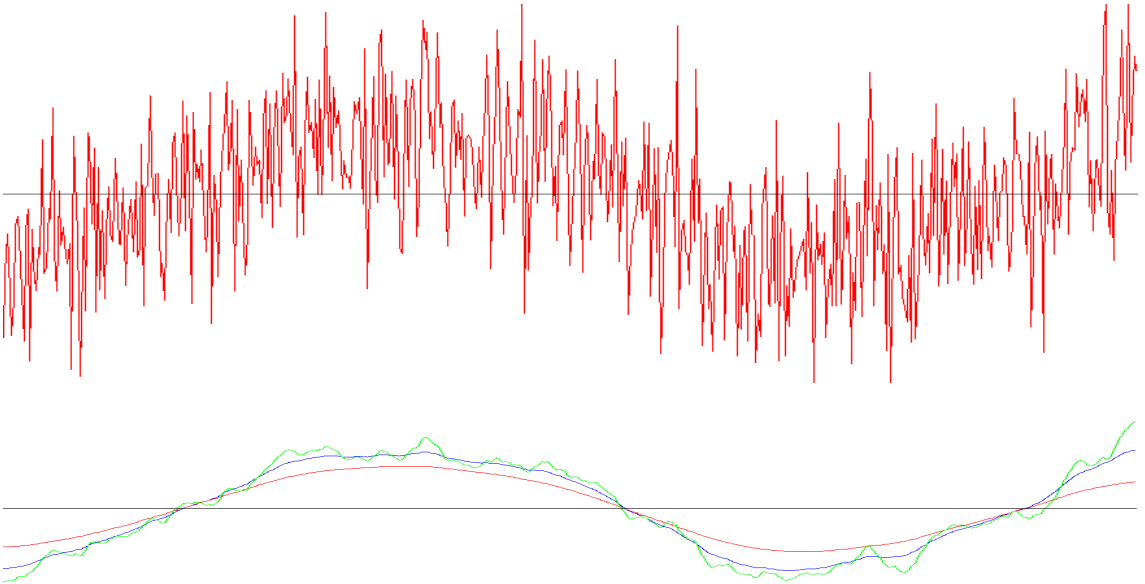
Dispersion of additive noise in the linear regression model depends on u:

$$\hat{D} = \hat{D}(u)$$

Criterion for choosing smoothness degree:

$$u^* = \arg\min_u \hat{D}(u)$$

Examples of processing with different smoothness degrees



Smoothing degree	Dispersion of additive noise
weak	100.53
optimal	97.81
excessive	103.16

Edge preserving smoothing

Edge functions $\gamma_t(\mathbf{x}_{t-1},\mathbf{x}_t)=(\mathbf{x}_{t-1}-\mathbf{A}\mathbf{x}_t)^T\mathbf{U}_t(\mathbf{x}_{t-1}-\mathbf{A}\mathbf{x}_t)$ **– a means of incorporating knowledge about presence of edges into the objective function**

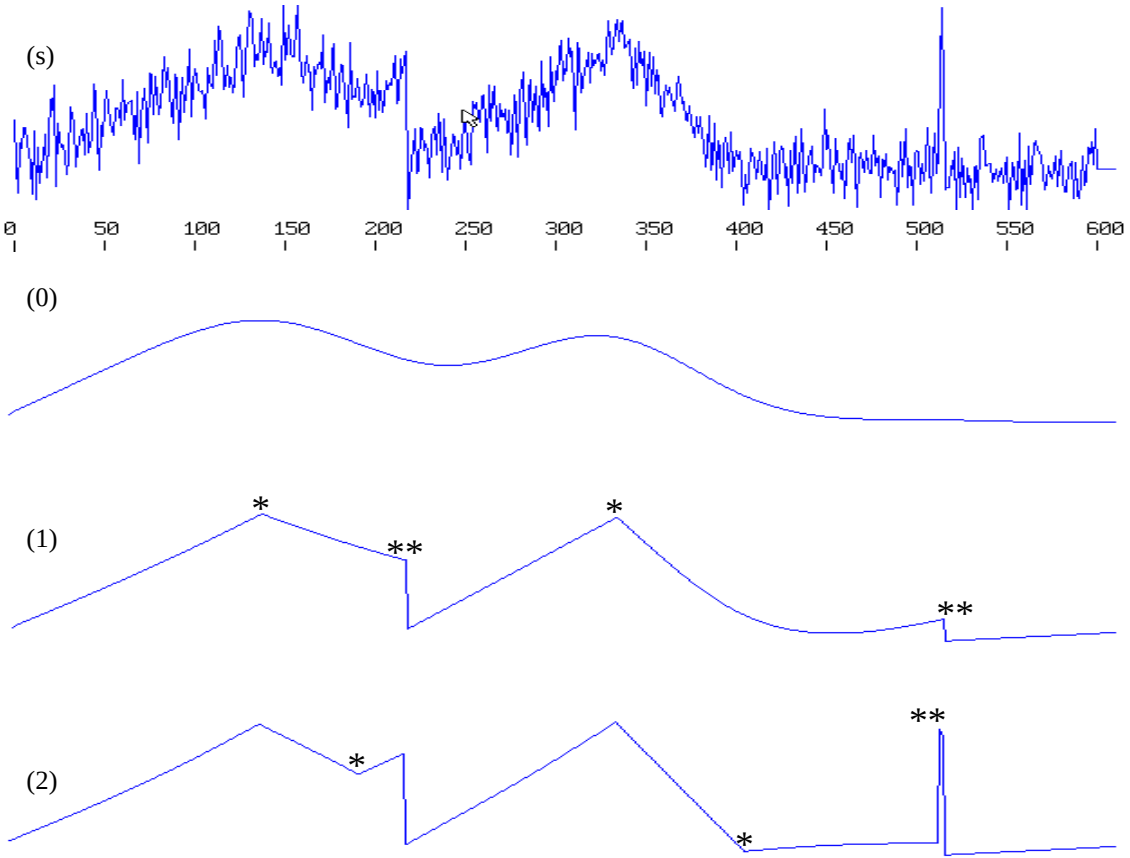
Criterion of edge estimation

$$z_t=\frac{\min\limits_{\mathbf{x}_1,\ldots,\mathbf{x}_N} J(\mathbf{x}_1,\ldots,\mathbf{x}_N)}{\min\limits_{\mathbf{x}_1,\ldots,\mathbf{x}_N} J^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)}=\frac{\min\limits_{\mathbf{x}_{t-1},\mathbf{x}_t}\left\{\tilde{J}_{t-1}^-(\mathbf{x}_{t-1})+\gamma_t(\mathbf{x}_{t-1},\mathbf{x}_t)+\tilde{J}_t^+(\mathbf{x}_t)\right\}}{\min\limits_{\mathbf{x}_{t-1},\mathbf{x}_t}\left\{\tilde{J}_{t-1}^-(\mathbf{x}_{t-1})+\gamma_t^*(\mathbf{x}_{t-1},\mathbf{x}_t)+\tilde{J}_t^+(\mathbf{x}_t)\right\}}$$

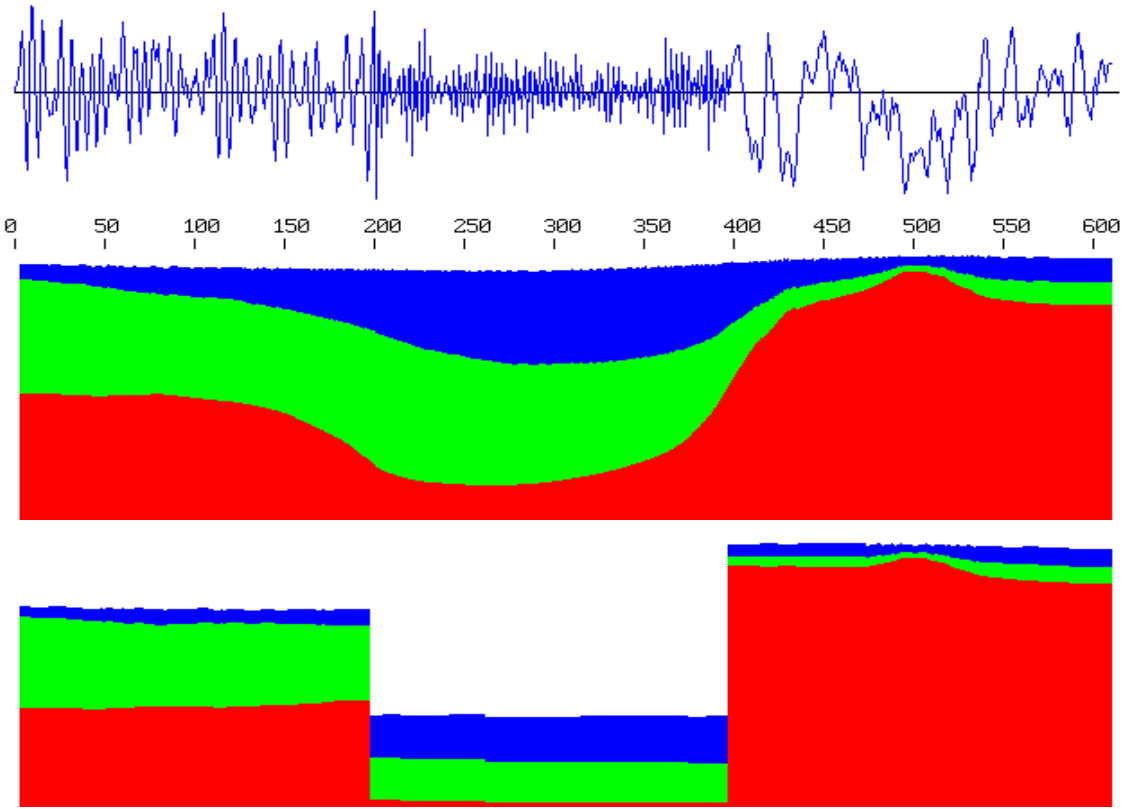
Criterion of edge estimation for a quadratic objective function

$$z_t(\mathbf{U}_t^*)=\frac{\tilde{b}_{t-1}^-+\tilde{b}_t^++(\tilde{\mathbf{x}}_{t-1}^--\tilde{\mathbf{x}}_t^+)^T\mathbf{U}_t\left[\mathbf{I}+\left((\tilde{\mathbf{Q}}_{t-1}^-)^{-1}+(\tilde{\mathbf{Q}}_t^+)^{-1}\right)\mathbf{U}_t\right]^{-1}(\tilde{\mathbf{x}}_{t-1}^--\tilde{\mathbf{x}}_t^+)}{\tilde{b}_{t-1}^-+\tilde{b}_t^++(\tilde{\mathbf{x}}_{t-1}^--\tilde{\mathbf{x}}_t^+)^T\mathbf{U}_t^*\left[\mathbf{I}+\left((\tilde{\mathbf{Q}}_{t-1}^-)^{-1}+(\tilde{\mathbf{Q}}_t^+)^{-1}\right)\mathbf{U}_t^*\right]^{-1}(\tilde{\mathbf{x}}_{t-1}^--\tilde{\mathbf{x}}_t^+)}$$

Example of edge preserving smoothing of a simulated signal



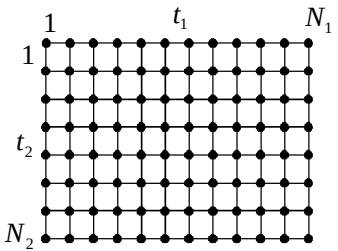
Edge preserving time-frequency analysis

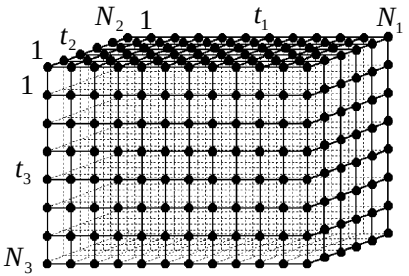


low, medium, and high frequency bands

A generalized procedure of multidimensional data analysis

Variable adjacency graphs for problems of image and 3D array analysis





Pairwise separable objective functions for images and 3D arrays

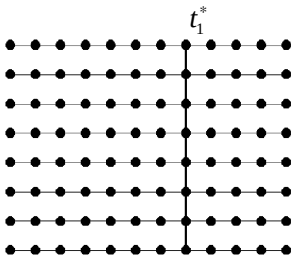
$$J(X)=\sum_{t_1=1}^{N_1}\sum_{t_2=1}^{N_2}\psi(\mathbf{x}_{t_1t_2})+\sum_{t_1=2}^{N_1}\sum_{t_2=1}^{N_2}\gamma'(\mathbf{x}_{t_1-1,t_2},\mathbf{x}_{t_1t_2})+\sum_{t_1=1}^{N_1}\sum_{t_2=2}^{N_2}\gamma''(\mathbf{x}_{t_1,t_2-1},\mathbf{x}_{t_1t_2})$$

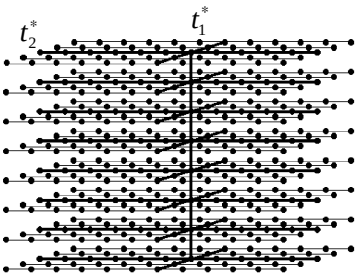
$$J(X)=\sum_{t_1=1}^{N_1}\sum_{t_2=1}^{N_2}\sum_{t_3=1}^{N_3}\psi(\mathbf{x}_{t_1t_2t_3})+\sum_{t_1=2}^{N_1}\sum_{t_2=1}^{N_2}\sum_{t_3=1}^{N_3}\gamma'(\mathbf{x}_{t_1-1,t_2t_3},\mathbf{x}_{t_1t_2t_3})+$$

$$\sum_{t_1=1}^{N_1}\sum_{t_2=2}^{N_2}\sum_{t_3=1}^{N_3}\gamma''(\mathbf{x}_{t_1,t_2-1,t_3},\mathbf{x}_{t_1t_2t_3})+\sum_{t_1=1}^{N_1}\sum_{t_2=1}^{N_2}\sum_{t_3=2}^{N_3}\gamma'''(\mathbf{x}_{t_1t_2,t_3-1},\mathbf{x}_{t_1t_2t_3})$$

Substitution of a lattice-like graph by a series of trees

Tree-like graphs for calculating values of variables in a vertical stem





Objective functions for one stem of an image and a 3D data array

$$J_{t_1^*}(X)=\sum_{t_1=1}^{N_1}\sum_{t_2=1}^{N_2}\psi(\mathbf{x}_{t_1t_2})+\sum_{t_2=2}^{N_2}\gamma''(\mathbf{x}_{t_1^*,t_2-1},\mathbf{x}_{t_1^*t_2})+\sum_{t_1=2}^{N_1}\sum_{t_2=1}^{N_2}\gamma'(\mathbf{x}_{t_1-1,t_2},\mathbf{x}_{t_1t_2}),\qquad t_1^*=1,\ldots,N_1$$

$$J_{t_1^*t_2^*}(X)=\sum_{t_1=1}^{N_1}\sum_{t_2=1}^{N_2}\sum_{t_3=1}^{N_3}\psi(\mathbf{x}_{t_1t_2t_3})+\sum_{t_3=2}^{N_3}\gamma'''(\mathbf{x}_{t_1^*t_2^*,t_3-1},\mathbf{x}_{t_1^*t_2^*t_3})+\sum_{t_1=2}^{N_1}\sum_{t_2=1}^{N_2}\sum_{t_3=1}^{N_3}\gamma'(\mathbf{x}_{t_1-1,t_2t_3},\mathbf{x}_{t_1t_2t_3})+$$

$$\sum_{t_1=1}^{N_1}\sum_{t_2=2}^{N_2}\sum_{t_3=1}^{N_3}\gamma''(\mathbf{x}_{t_1,t_2-1,t_3},\mathbf{x}_{t_1t_2t_3}),\qquad t_1^*=1,\ldots,N_1,\ \ t_2^*=1,\ldots,N_2$$

Generalized procedure of image processing

Horizontal processing:

– optimization procedure "forward and back" or "forward against forward" is independently applied to each horizontal branch $(1t_2,...,N_1 t_2)$, $t_2 = 1,...,N_2$.

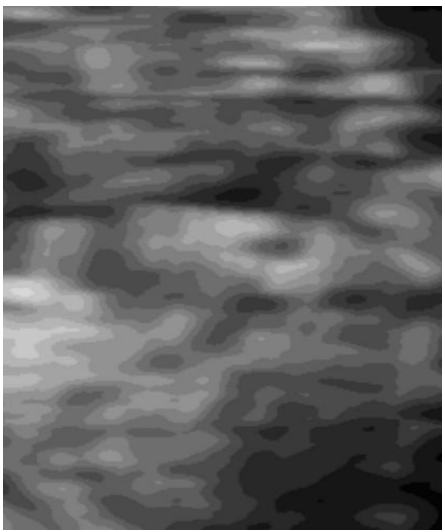
– marginal node functions are stored at each node.

Vertical processing:

– the same optimization procedure is independently applied to each vertical branch $(t_1 1,...,t_1 N_2)$, $t_1 = 1,...,N_1$, but marginal node functions $\hat{J}_{t_1 t_2}(\mathbf{x}_{t_1 t_2})$ are used instead of node functions $\psi_{t_1 t_2}(\mathbf{x}_{t_1 t_2})$ at each node.

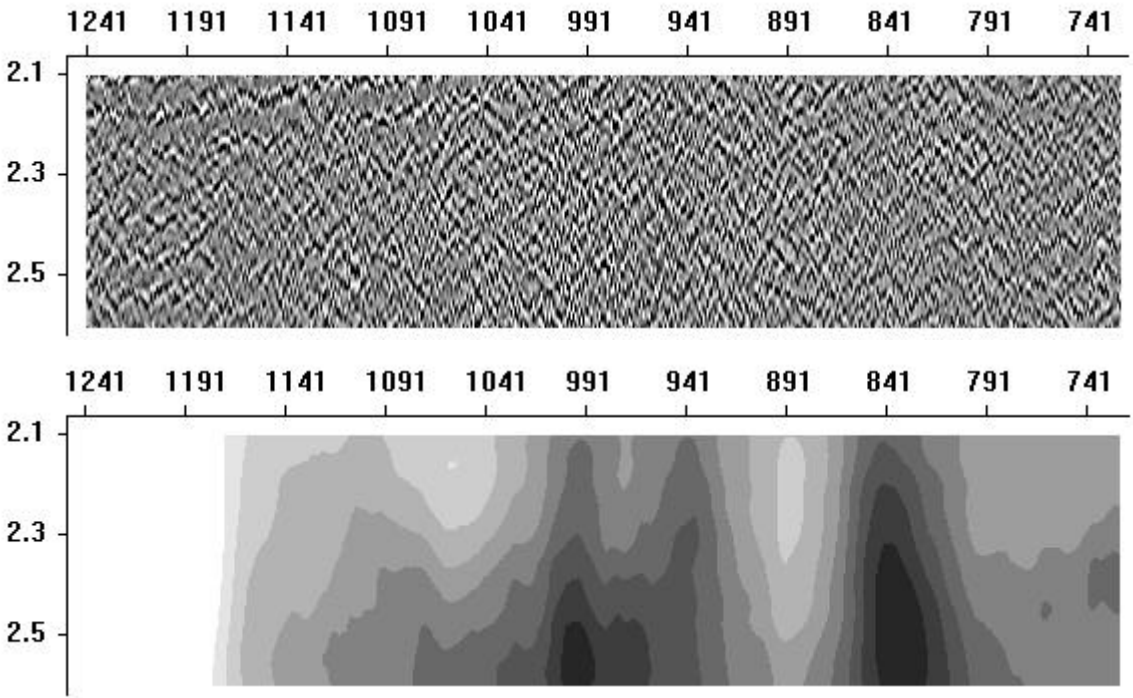
Image enhancement problem

A satellite photograph of the Earth surface, the smoothed image, and result of brightness correction by subtracting the smoothed image from the original.

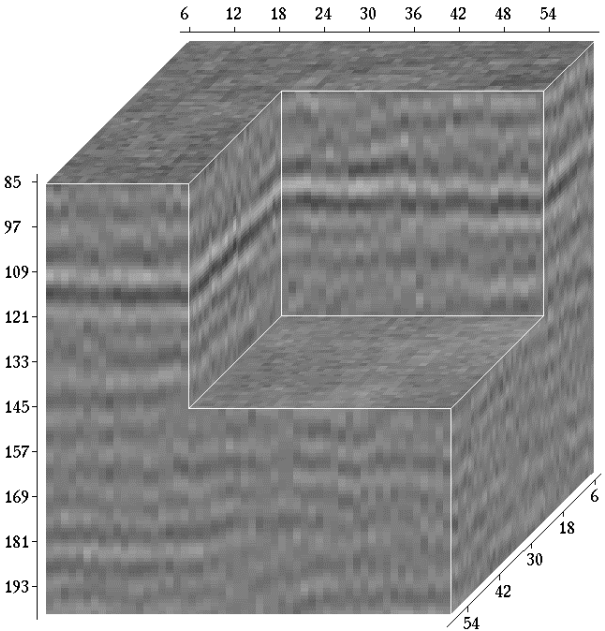


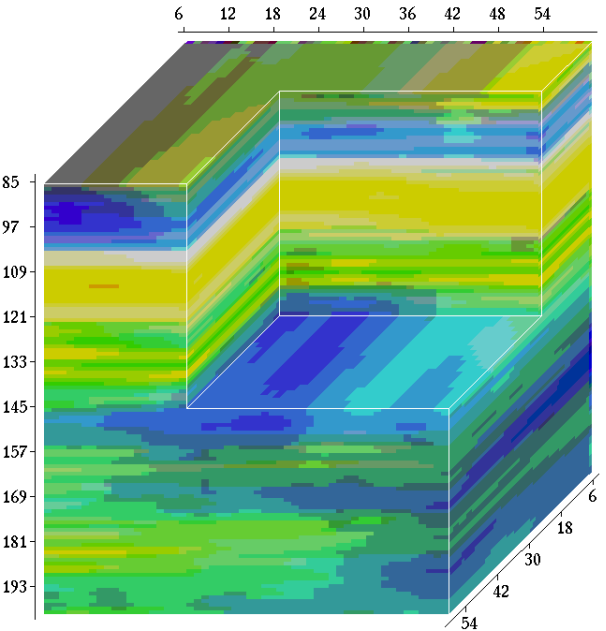
Texture analysis of images

Fragment of a seismic section and result of its spatial auturegression analysis represented as local signal oscillation intensity in high frequency band in the horizontal direction



Texture analysis of 3D images





Directional representation of fingerprints